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Coupled coincidences for multi-valued contractions in partially ordered metric spaces

N Hussain and A Alotaibi*

* Correspondence: aalotaibi@kau.edu.sa
Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Abstract

In this article, we study the existence of coupled coincidence points for multi-valued nonlinear contractions in partially ordered metric spaces. We do it from two different approaches, the first is Δ-symmetric property recently studied in Samet and Vetro (Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces, Nonlinear Anal. **74**, 4260-4268 (2011)) and second one is mixed *g*-monotone property studied by Lakshmikantham and Ćirić (Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. **70**, 4341-4349 (2009)).

The theorems presented extend certain results due to Ćirić (Multi-valued nonlinear contraction mappings, Nonlinear Anal. **71**, 2716-2723 (2009)), Samet and Vetro (Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces, Nonlinear Anal. **74**, 4260-4268 (2011)) and many others. We support the results by establishing an illustrative example.

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1. Introduction and preliminaries

Let (X, d) be a metric space. We denote by $CB(X)$ the collection of non-empty closed bounded subsets of X. For $A, B \in CB(X)$ and $x \in X$, suppose that

$$D(x, A) = \inf_{a\in A} d(x, a) \quad \text{and} \quad H(A, B,) = \max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\}.$$

Such a mapping H is called a Hausdorff metric on $CB(X)$ induced by d.

Definition 1.1. *An element $x \in X$ is said to be a fixed point of a multi-valued mapping $T: X \to CB(X)$ if and only if $x \in Tx$.*

In 1969, Nadler [1] extended the famous Banach Contraction Principle from single-valued mapping to multi-valued mapping and proved the following fixed point theorem for the multi-valued contraction.

Theorem 1.1. *Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $c \in [0,1)$ such that $H(Tx, Ty) \le cd(x, y)$ for all $x,y \in X$. Then, T has a fixed point.*

The existence of fixed points for various multi-valued contractive mappings has been studied by many authors under different conditions. In 1989, Mizoguchi and Takahashi [2] proved the following interesting fixed point theorem for a weak contraction.

Theorem 1.2. *Let* (X,d) *be a complete metric space and let* T *be a mapping from* X *into* $CB(X)$. *Assume that* $H(Tx, Ty) \leq \alpha(d(x,y))\, d(x,y)$ *for all* $x,y \in X$, *where* α *is a function from* $[0,\infty)$ *into* $[0,1)$ *satisfying the condition* $\limsup_{s\to t^+}\alpha(s) < 1$ *for all* $t \in [0,\infty)$. *Then,* T *has a fixed point.*

Let $CL(X) := \{A \subset X | A \neq \Phi, \bar{A} = A\}$, where $\bar{A}$ denotes the closure of A in the metric space (X, d). In this context, Ćirić [3] proved the following interesting theorem.

Theorem 1.3. *(See* [3]*) Let* (X,d) *be a complete metric space and let* T *be a mapping from* X *into* $CL(X)$. *Let* $f: X \to \mathbb{R}$ *be the function defined by* $f(x) = d(x, Tx)$ *for all* $x \in X$. *Suppose that* f *is lower semi-continuous and that there exists a function* $\varphi: [0, +\infty) \to [a, 1)$, $0 < a < 1$, *satisfying*

$$\limsup_{r\to t^+}\phi(r) < 1 \quad \text{for each } t \in [0,+\infty). \tag{1.1}$$

Assume that for any $x \in X$ *there is* $y \in Tx$ *satisfying the following two conditions:*

$$\sqrt{\phi(f(x))}\,d(x,y) \leq f(x) \tag{1.2}$$

such that

$$f(y) \leq \phi(f(x))d(x,y). \tag{1.3}$$

Then, there exists $z \in X$ *such that* $z \in Tz$.

Definition 1.2. [4]*Let* X *be a non-empty set and* $F: X \times X \to X$ *be a given mapping. An element* $(x, y) \in X \times X$ *is said to be a coupled fixed point of the mapping* F *if* $F(x, y) = x$ *and* $F(y, x) = y$.

Definition 1.3. [5]*Let* $(x,y) \in X \times X$, $F: X \times X \to X$ *and* $g: X \to X$. *We say that* (x,y) *is a coupled coincidence point of* F *and* g *if* $F(x,y) = gx$ *and* $F(y, x) = gy$ *for* $x,y \in X$.

Definition 1.4. *A function* $f: X \times X \to \mathbb{R}$ *is called lower semi-continuous if and only if for any sequence* $\{x_n\} \subset X$, $\{y_n\} \subset X$ *and* $(x,y) \in X \times X$, *we have*

$$\lim_{n\to\infty}(x_n, y_n) = (x, y) \Rightarrow f(x,y) \leq \liminf_{n\to\infty} f(x_n, y_n).$$

Let (X, d) be a metric space endowed with a partial order and $G: X \to X$ be a given mapping. We define the set $\Delta \subset X \times X$ by

$$\Delta := \{(x,y) \in X \times X | G(x) \preccurlyeq G(y)\}.$$

In [6], Samet and Vetro introduced the binary relation R on $CL(X)$ defined by

$$ARB \Leftrightarrow A \times B \subseteq \Delta,$$

where $A, B \in CL(X)$.

Definition 1.5. *Let* $F: X \times X \to CL(X)$ *be a given mapping. We say that* F *is a* Δ*-symmetric mapping if and only if* $(x,y) \in \Delta \Rightarrow F(x,y)RF(y,x)$.

Example 1.1. *Suppose that* $X = [0,1]$, *endowed with the usual order* $\leq$. *Let* $G: [0,1] \to [0,1]$ *be the mapping defined by* $G(x) = M$ *for all* $x \in [0,1]$, *where* M *is a constant in* $[0,1]$. *Then,* $\Delta = [0,1] \times [0,1]$ *and* F *is a* Δ*-symmetric mapping.*

Definition 1.6. [6]*Let* $F: X \times X \to CL(X)$ *be a given mapping. We say that* $(x,y) \in X \times X$ *is a coupled fixed point of* F *if and only if* $x \in F(x,y)$ *and* $y \in F(y,x)$.

Definition 1.7. *Let F: $X \times X \to CL(X)$ be a given mapping and let g: $X \to X$. We say that $(x,y) \in X \times X$ is a coupled coincidence point of F and g if and only if $gx \in F(x,y)$ and $gy \in F(y,x)$.*

In [6], Samet and Vetro proved the following coupled fixed point version of Theorem 1.3.

Theorem 1.4. *Let (X, d) be a complete metric space endowed with a partial order $\preccurlyeq$. We assume that $\Delta \neq \emptyset$, i.e., there exists $(x_0,y_0) \in \Delta$. Let F: $X \times X \to CL(X)$ be a Δ-symmetric mapping. Suppose that the function f: $X \times X \to [0,+\infty)$ defined by*

$$f(x,y) := D(x,F(x,y)) + D(y,F(y,x)) \quad \text{for all } x,y \in X$$

is lower semi-continuous and that there exists a function φ: $[0, \infty) \to [a, 1)$, $0 < a < 1$, satisfying

$$\limsup_{r \to t^+} \phi(r) < 1 \quad \text{for each } t \in [0,+\infty).$$

Assume that for any $(x,y) \in \Delta$ there exist $u \in F(x,y)$ and $v \in F(y,x)$ satisfying

$$\sqrt{\phi(f(x,y))}[d(x,u) + d(y,v)] \leq f(x,y)$$

such that

$$f(u,v) \leq \phi(f(x,y))[d(x,u) + d(y,v)].$$

Then, F admits a coupled fixed point, i.e., there exists $z = (z_1, z_2) \in X \times X$ such that $z_1 \in F(z_1, z_2)$ and $z_2 \in F(z_2, z_1)$.

In 2006, Bhaskar and Lakshmikantham [4] introduced the notion of a coupled fixed point and established some coupled fixed point theorems in partially ordered metric spaces. They have discussed the existence and uniqueness of a solution for a periodic boundary value problem. Lakshmikantham and Ćirić [5] proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces using mixed g-monotone property. For more details on coupled fixed point theory, we refer the reader to [7-12] and the references therein. Here we study the existence of coupled coincidences for multi-valued nonlinear contractions using two different approaches, first is based on Δ-symmetric property recently studied in [6] and second one is based on mixed g-monotone property studied by Lakshmikantham and Ćirić [5]. The theorems presented extend certain results due to Ćirić [3], Samet and Vetro [6] and many others. We support the results by establishing an illustrative example.

2. Coupled coincidences by Δ-symmetric property

Following is the main result of this section which generalizes the above mentioned results of Ćirić, and Samet and Vetro.

Theorem 2.1. *Let (X,d) be a metric space endowed with a partial order $\preccurlyeq$ and $\Delta \neq \emptyset$. Suppose that F: $X \times X \to CL(X)$ is a Δ-symmetric mapping, g: $X \to X$ is continuous, gX is complete, the function f: $g(X) \times g(X) \to [0, +\infty)$ defined by*

$$f(gx,gy) := D(gx,F(x,y)) + D(gy,F(y,x)) \quad \text{for all } x,y \in X$$

is lower semi-continuous and that there exists a function φ: $[0, \infty) \to [a, 1)$, $0 < a < 1$, *satisfying*

$$\limsup_{r \to t^+} \phi(r) < 1 \quad \textit{for each } t \in [0, +\infty). \tag{2.1}$$

Assume that for any $(x,y) \in \Delta$ *there exist* $gu \in F(x,y)$ *and* $gv \in F(y,x)$ *satisfying*

$$\sqrt{\phi(f(gx, gy))}[d(gx, gu) + d(gy, gv)] \le f(gx, gy) \tag{2.2}$$

such that

$$f(gu, gv) \le \phi(f(gx, gy))[d(gx, gu) + d(gy, gv)]. \tag{2.3}$$

Then, F and g have a coupled coincidence point, i.e., there exists $gz = (gz_1, gz_2) \in X \times X$ *such that* $gz_1 \in F(z_1, z_2)$ *and* $gz_2 \in F(z_2, z_1)$.

Proof. Since by the definition of φ we have $\varphi(f(x,y)) < 1$ for each $(x,y) \in X \times X$, it follows that for any $(x,y) \in X \times X$ there exist $gu \in F(x,y)$ and $gv \in F(y,x)$ such that

$$\sqrt{\phi(f(gx, gy))}d(gx, gu) \le D(gx, F(x, y))$$

and

$$\sqrt{\phi(f(gx, gy))}d(gy, gv) \le D(gy, F(y, x)).$$

Hence, for each $(x,y) \in X \times X$, there exist $gu \in F(x,y)$ and $gv \in F(y,x)$ satisfying (2.2).

Let $(x_0, y_0) \in \Delta$ be arbitrary and fixed. By (2.2) and (2.3), we can choose $gx_1 \in F(x_0, y_0)$ and $gy_1 \in F(y_0, x_0)$ such that

$$\sqrt{\phi(f(gx_0, gy_0))}[d(gx_0, gx_1) + d(gy_0, gy_1)] \le f(gx_0, gy_0) \tag{2.4}$$

and

$$f(gx_1, gy_1) \le \phi(f(gx_0, gy_0))[d(gx_0, gx_1) + d(gy_0, gy_1)]. \tag{2.5}$$

From (2.4) and (2.5), we can get

$$\begin{aligned} f(gx_1, gy_1) &\le \phi(f(gx_0, gy_0))[d(gx_0, gx_1) + d(gy_0, gy_1)] \\ &= \sqrt{\phi(f(gx_0, gy_0))}\{\sqrt{\phi(f(gx_0, gy_0))}[d(gx_0, gx_1) + d(gy_0, gy_1)]\} \\ &\le \sqrt{\phi(f(gx_0, gy_0))}f(gx_0, gy_0). \end{aligned}$$

Thus,

$$f(gx_1, gy_1) \le \sqrt{\phi(f(gx_0, gy_0))}f(gx_0, gy_0). \tag{2.6}$$

Now, since F is a Δ-symmetric mapping and $(x_0, y_0) \in \Delta$, we have

$$F(x_0, y_0)RF(y_0, x_0) \Rightarrow (x_1, y_1) \in \Delta.$$

Also, by (2.2) and (2.3), we can choose $gx_2 \in F(x_1, y_1)$ and $gy_2 \in F(y_1, x_1)$ such that

$$\sqrt{\phi(f(gx_1, gy_1))}[d(gx_1, gx_2) + d(gy_1, gy_2)] \le f(gx_1, gy_1)$$

and

$$f(gx_2, gy_2) \le \phi(f(gx_1, gy_1))[d(gx_1, gx_2) + d(gy_1, gy_2)].$$

Hence, we get

$$f(gx_2 gy_2) \leq \sqrt{\phi(f(gx_1, gy_1))} f(gx_1, gy_1),$$

with $(x_2, y_2) \in \Delta$.

Continuing this process we can choose $\{gx_n\} \subset X$ and $\{gy_n\} \subset X$ such that for all $n \in \mathbb{N}$, we have

$$(x_n, y_n) \in \Delta, \quad gx_{n+1} \in F(x_n, y_n), \quad gy_{n+1} \in F(y_n, x_n), \tag{2.7}$$

$$\sqrt{\phi(f(gx_n, gy_n))}[d(gx_n, gx_{x+1}) + d(gy_n, gy_{n+1})] \leq f(gx_n, gy_n), \tag{2.8}$$

and

$$f(gx_{n+1}, gy_{n+1}) \leq \sqrt{\phi(f(gx_n, gy_n))} f(gx_n, gy_n). \tag{2.9}$$

Now, we shall show that $f(gx_n, gy_n) \to 0$ as $n \to \infty$. We shall assume that $f(gx_n, gy_n) > 0$ for all $n \in \mathbb{N}$, since if $f(gx_n, gy_n) = 0$ for some $n \in \mathbb{N}$, then we get $D(gx_n, F(x_n, y_n)) = 0$ which implies that $gx_n \in \overline{F(x_n, y_n)} = F(x_n, y_n)$ and $D(gy_n, F(y_n, x_n)) = 0$ which implies that $gy_n \in F(y_n, x_n)$. Hence, in this case, (x_n, y_n) is a coupled coincidence point of F and g and the assertion of the theorem is proved.

From (2.9) and $\varphi(t) < 1$, we deduce that $\{f(gx_n, gy_n)\}$ is a strictly decreasing sequence of positive real numbers. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n\to\infty} f(gx_n, gy_n) = \delta.$$

Now, we will prove that $\delta = 0$. Suppose that this is not the case; taking the limit on both sides of (2.9) and having in mind the assumption (2.1), we have

$$\delta \leq \limsup_{f(gx_n, gy_n)\to\delta^+} \sqrt{\phi(f(gx_n, gy_n))}\delta < \delta,$$

a contradiction. Thus, $\delta = 0$, that is,

$$\lim_{n\to\infty} f(gx_n, gy_n) = 0. \tag{2.10}$$

Now, let us prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in (X, d). Suppose that

$$\alpha = \limsup_{f(gx_n, gy_n)\to 0^+} \sqrt{\phi(f(gx_n, gy_n))}.$$

Then, by assumption (2.1), we have $\alpha < 1$. Let q be such that $\alpha < q < 1$. Then, there is some $n_0 \in \mathbb{N}$ such that

$$\sqrt{\phi(f(gx_n, gy_n))} < q \quad \text{for each } n \geq n_0.$$

Thus, from (2.9), we get

$$f(gx_{n+1}, gy_{n+1}) \leq q f(gx_n, gy_n) \quad \text{for each } n \geq n_0.$$

Hence, by induction,

$$f(gx_{n+1}, gy_{n+1}) \leq q^{n+1-n_0} f(gx_{n0}, gy_{n_0}) \quad \text{for each } n \geq n_0. \tag{2.11}$$

Since $\varphi(t) \geq a > 0$ for all $t \geq 0$, from (2.8) and (2.11), we obtain

$$d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) \leq \frac{1}{\sqrt{a}} q^{n-n_0} f(gx_{n_0}, gy_{n_0}) \quad \text{for each } n \geq n_0. \tag{2.12}$$

From (2.12) and since $q < 1$, we conclude that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in (X,d).

Now, since gX is complete, there is a $w = (w_1, w_2) \in gX \times gX$ such that

$$\lim_{n\to\infty} gx_n = w_1 = gz_1 \quad \text{and} \quad \lim_{n\to\infty} gy_n = w_2 = gz_2 \tag{2.13}$$

for some z_1, z_2 in X. We now show that $z = (z_1, z_2)$ is a coupled coincidence point of F and g. Since by assumption f is lower semi-continuous so from (2.10), we get

$$0 \leq f(gz_1, gz_2) = D(gz_1, F(z_1, z_2)) + D(gz_2, F(z_2, z_1)) \leq \liminf_{n\to\infty} f(gx_n, gy_n) = 0.$$

Hence,

$$D(gz_1, F(z_1, z_2)) = D(gz_2, F(z_2, z_1)) = 0,$$

which implies that $gz_1 \in F(z_1, z_2)$ and $gz_2 \in F(z_2, z_1)$, i.e., $z = (z_1, z_2)$ is a coupled coincidence point of F and g. This completes the proof.

Now, we prove the following theorem.

Theorem 2.2. *Let (X, d) be a metric space endowed with a partial order $\preccurlyeq$ and $\Delta \neq \emptyset$. Suppose that F: $X \times X \to CL(X)$ is a Δ-symmetric mapping, g: $X \to X$* is *continuous and gX is complete. Suppose that the function f: $gX \times gX \to [0,+\infty)$ defined in Theorem 2.1 is lower semi-continuous and that there exists a function φ: $[0, +\infty) \to [a, 1)$, $0 < a < 1$, satisfying*

$$\limsup_{r\to t+} \phi(r) < 1 \quad \textit{for each } t \in [0,\infty). \tag{2.14}$$

Assume that for any $(x,y) \in \Delta$, there exist $gu \in F(x,y)$ and $gv \in F(y,x)$ satisfying

$$\sqrt{\phi(d(gx, gu) + d(gy, gv))}[d(gx, gu) + d(gy, gv)] \leq D(gx, F(x, y)) + D(gy, F(y, x)) \tag{2.15}$$

such that

$$D(gu, F(u, v)) + D(gv, F(v, u)) \leq \phi(d(gx, gu) + d(gy, gv))[d(gx, gu) + d(gy, gv)]. \tag{2.16}$$

Then, F and g have a coupled coincidence point, i.e., there exists $z = (z_1, z_2) \in X \times X$ such that $gz_1 \in F(z_1, z_2)$ and $gz_2 \in F(z_2, z_1)$.

Proof. Replacing $\varphi(f(x,y))$ with $\varphi(d(gx, gu) + d(gy, gv))$ and following the lines in the proof of Theorem 2.1, one can construct iterative sequences $\{x_n\} \subset X$ and $\{y_n\} \subset X$ such that for all $n \in \mathbb{N}$, we have

$$(x_n, y_n) \in \Delta, \quad gx_{n+1} \in F(x_n, y_n), \quad gy_{n+1} \in F(y_n, x_n), \tag{2.17}$$

$$\begin{aligned} &\sqrt{\phi(d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}))}[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})] \\ &\leq D(gx_n, F(x_n, y_n)) + D(gy_n, F(y_n, x_n)) \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} & D(gx_{n+1}, F(x_{n+1}, y_{n+1})) + D(gy_{n+1}, F(y_{n+1}, x_{n+1})) \\ & \leq \sqrt{\varphi(d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}))}[D(gx_n, F(x_n, y_n)) + D(gy_n, F(y_n, x_n))] \end{aligned} \tag{2.19}$$

for all $n \geq 0$. Again, following the lines of the proof of Theorem 2.1, we conclude that $\{D(gx_n, F(x_n, y_n)) + D(gy_n, F(y_n, x_n))\}$ is a strictly decreasing sequence of positive real numbers. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n\to+\infty} \{D(gx_n, F(x_n, y_n)) + D(gy_n, F(y_n, x_n)) = \delta. \tag{2.20}$$

Since in our assumptions there appears φ $(d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}))$, we need to prove that $\{d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})\}$ admits a subsequence converging to a certain η^+ for some $\eta \geq 0$. Since $\phi(t) \geq a > 0$ for all t ≥ 0, from (2.18) we obtain

$$d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) \leq \frac{1}{\sqrt{a}}[D(gx_n, F(x_n, y_n)) + D(gy_n, F(y_n, x_n))]. \tag{2.21}$$

From (2.20) and (2.21), we conclude that the sequence $\{d(gx_n, gx_{n+1})+d(gy_n, gy_{n+1})\}$ is bounded. Therefore, there is some $\theta \geq 0$ such that

$$\liminf_{n\to+\infty}\{d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})\} = \theta. \tag{2.22}$$

Since $gx_{n+1} \in F(x_n, y_n)$ and $gy_{n+1} \in F(y_n, x_n)$, it follows that

$$d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) \geq D(gx_n, F(x_n, y_n)) + D(gy_n, F(y_n, x_n))$$

for each $n \geq 0$. This implies that $\theta \geq \delta$. Now, we shall show that $\theta = \delta$. If we assume that $\delta = 0$, then from (2.20) and (2.21) we have

$$\lim_{n\to+\infty}\{d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})\} = 0.$$

Thus, if $\delta = 0$, then $\theta = \delta$. Suppose now that $\delta > 0$ and suppose, to the contrary, that $\theta > \delta$. Then, $\theta - \delta > 0$ and so from (2.20) and (2.22) there is a positive integer n_0 such that

$$D(gx_n, F(x_n, y_n)) + D(gy_n, F(y_n, x_n)) < \delta + \frac{\theta - \delta}{4} \tag{2.23}$$

and

$$\theta - \frac{\theta - \delta}{4} < d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \tag{2.24}$$

for all $n \geq n_0$. Then, combining (2.18), (2.23) and (2.24) we get

$$\begin{aligned} & \sqrt{\varphi(d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}))}\left(\theta - \frac{\theta - \delta}{4}\right) \\ & < \sqrt{\varphi(d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}))}[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})] \\ & \leq D(gx_n, F(x_n, y_n)) + D(gy_n, F(y_n, x_n)) \\ & < \delta + \frac{\theta - \delta}{4} \end{aligned}$$

for all $n \geq n_0$. Hence, we get

$$\sqrt{\varphi(d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}))} \leq \frac{\theta + 3\delta}{3\theta + \delta} \tag{2.25}$$

for all $n \geq n_0$. Set $h = \dfrac{\theta + 3\delta}{3\theta + \delta} < 1$. Now, from (2.19) and (2.25), it follows that

$$D(gx_{n+1}, F(x_{n+1}, y_{n+1})) + D(gy_{n+1}, F(y_{n+1}, x_{n+1})) \leq h[D(gx_n, F(x_n, y_n)) + D(gy_n, F(y_n, x_n))]$$

for all $n \geq n_0$. Finally, since we assume that $\delta > 0$ and as $h < 1$, proceeding by induction and combining the above inequalities, it follows that

$$\begin{aligned}\delta &\leq D(gx_{n_0+k_0}, F(x_{n_0+k_0}, y_{n_0+k_0})) + D(gy_{n_0+k_0}, F(y_{n_0+k_0}, x_{n_0+k_0}))\\ &\leq h^{k_0} D(gx_{n_0}, F(x_{n_0}, y_{n_0})) + D(gy_{n_0}, F(y_{n_0}, x_{n_0})) < \delta\end{aligned}$$

for a positive integer k_0, which is a contradiction to the assumption $\theta > \delta$ and so we must have $\theta = \delta$. Now, we shall show that $\theta = 0$. Since

$$\theta = \delta \leq D(gx_n, F(x_n, y_n)) + D(gy_n, F(y_n, x_n)) \leq d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}),$$

so we can read (2.22) as

$$\liminf_{n \to +\infty} \{d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})\} = \theta^+.$$

Thus, there exists a subsequence $\{d(gx_{n_k}, gx_{n_k+1}) + d(gy_{n_k}, gy_{n_k+1})\}$ such that

$$\lim_{k \to +\infty} \{d(gx_{n_k}, gx_{n_k+1}) + d(gy_{n_k}, gy_{n_k+1})\} = \theta^+.$$

Now, by (2.14), we have

$$\limsup_{(d(gx_{n_k}, gx_{n_k+1}) + d(gy_{n_k}, gy_{n_k+1})) \to \theta^+} \sqrt{\varphi(d(gx_{n_k}, gx_{n_k+1}) + d(gy_{n_k}, gy_{n_k+1}))} < 1. \tag{2.26}$$

From (2.19),

$$\begin{aligned}&D(gx_{n_k+1}, F(x_{n_k+1}, y_{n_k+1})) + D(gy_{n_k+1}, F(y_{n_k+1}, x_{n_k+1}))\\ &\leq \sqrt{\varphi(d(gx_{n_k}, gx_{n_k+1}) + d(gy_{n_k}, gy_{n_k+1}))}[D(gx_{n_k}, F(x_{n_k}, y_{n_k})) + D(gy_{n_k}, F(y_{n_k}, x_{n_k}))].\end{aligned}$$

Taking the limit as $k \to +\infty$ and using (2.20), we get

$$\begin{aligned}\delta &= \limsup_{k \to +\infty} \{D(gx_{n_k+1}, F(x_{n_k+1}, y_{n_k+1})) + D(gy_{n_k+1}, F(y_{n_k+1}, x_{n_k+1}))\}\\ &\leq \left(\limsup_{k \to +\infty} \sqrt{\varphi(d(gx_{n_k}, gx_{n_k+1}) + d(gy_{n_k}, gy_{n_k+1}))}\right)\\ &\quad (\limsup_{k \to +\infty} \{D(gx_{n_k}, F(x_{n_k}, y_{n_k})) + D(gy_{n_k}, F(y_{n_k}, x_{n_k}))\})\\ &= \left(\limsup_{(d(gx_{n_k}, gx_{n_k+1}) + d(gy_{n_k}, gy_{n_k+1})) \to \theta^+} \sqrt{\varphi(d(gx_{n_k}, gx_{n_k+1}) + d(gy_{n_k}, gy_{n_k+1}))}\right) \delta.\end{aligned}$$

From the last inequality, if we suppose that $\delta > 0$, we get

$$1 \leq \limsup_{(d(gx_{n_k}, gx_{n_k+1}) + d(gy_{n_k}, gy_{n_k+1})) \to \theta+} \sqrt{\varphi(d(gx_{n_k}, gx_{n_k+1}) + d(gy_{n_k}, gy_{n_k+1}))},$$

a contradiction with (2.26). Thus, $\delta = 0$. Then, from (2.20) and (2.21) we have

$$\alpha = \limsup_{(d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1}))\to 0+} \sqrt{\varphi(d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1}))} < 1.$$

Once again, proceeding as in the proof of Theorem 2.1, one can prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in gX and that $z = (z_1, z_2) \in X \times X$ is a coupled coincidence point of F, g, i.e.

$$gz_1 \in F(z_1,z_2) \quad \text{and} \quad gz_2 \in F(z_2,z_1).$$

Example 2.3. Suppose that $X = [0,1]$, equipped with the usual metric d: $X \times X \to [0, +\infty)$, and G: $[0,1] \to [0,1]$ is the mapping defined by

$$G(x) = M \quad \text{for all } x \in [0,1],$$

where M is a constant in [0,1]. Let F: $X \times X \to CL(X)$ be defined as

$$F(x,y) = \begin{cases} \frac{x^2}{4} & \text{if } y \in [0,\frac{15}{32}) \cup (\frac{15}{32},1], \\ \{\frac{15}{96},\frac{1}{5}\} & \text{if } y = \frac{15}{32}. \end{cases}$$

Then, $\Delta = [0,1] \times [0,1]$ and F is a Δ-symmetric mapping. Define now ϕ: $[0, +\infty) \to [0,1)$ by

$$\varphi(t) = \begin{cases} \frac{11}{12}t & \text{if } t \in [0,\frac{2}{3}], \\ \frac{11}{18} & \text{if } t \in (\frac{2}{3},+\infty). \end{cases}$$

Let g: $[0,1] \to [0,1]$ be defined as $gx = x^2$. Now, we shall show that $F(x, y)$ satisfies all the assumptions of Theorem 2.2. Let

$$f(x,y) = \begin{cases} \sqrt{x}+\sqrt{y}-\frac{1}{4}(x+y) & \text{if } x,y \in [0,\frac{15}{32}) \cup (\frac{15}{32},1], \\ \sqrt{x}-\frac{1}{4}x+\frac{43}{160} & \text{if } x \in [0,\frac{15}{32}) \cup (\frac{15}{32},1] \text{ and } y = \frac{15}{32}, \\ \sqrt{y}-\frac{1}{4}y+\frac{43}{160} & \text{if } y \in [0,\frac{15}{32}) \cup (\frac{15}{32},1] \text{ and } x = \frac{15}{32}, \\ \frac{43}{80} & \text{if } x = y = \frac{15}{32}. \end{cases}$$

It is easy to see that the function

$$f(gx,gy) = \begin{cases} x+y-\frac{1}{4}(x^2+y^2) & \text{if } x,y \in [0,\frac{15}{32}) \cup (\frac{15}{32},1], \\ x-\frac{1}{4}x^2+\frac{43}{160} & \text{if } x \in [0,\frac{15}{32}) \cup (\frac{15}{32},1] \text{ and } y = \frac{15}{32}, \\ y-\frac{1}{4}y^2+\frac{43}{160} & \text{if } y \in [0,\frac{15}{32}) \cup (\frac{15}{32},1] \text{ and } x = \frac{15}{32}, \\ \frac{43}{80} & \text{if } x = y = \frac{15}{32} \end{cases}$$

is lower semi-continuous. Therefore, for all x, $y \in [0,1]$ with $x, y \neq \frac{15}{32}$, there exist $gu \in F(x,y) = \{\frac{x^2}{4}\}$ and $gv \in F(y,x) = \{\frac{y^2}{4}\}$ such that

$$\begin{aligned} D(gu,F(u,v)) + D(gv,F(v,u)) &= \frac{x^2}{4} - \frac{x^4}{64} + \frac{y^2}{4} - \frac{y^4}{64} \\ &= \frac{1}{4}\left[\left(x+\frac{x^2}{4}\right)\left(x-\frac{x^2}{4}\right) + \left(y+\frac{y^2}{4}\right)\left(y-\frac{y^2}{4}\right)\right] \\ &\le \frac{1}{4}\left[\left(x+\frac{x^2}{4}\right)d(gx,gu) + \left(y+\frac{y^2}{4}\right)d(gy,gv)\right] \\ &\le \frac{1}{2}\max\left\{x+\frac{x^2}{4}, y+\frac{y^2}{4}\right\}[d(gx,gu)+d(gy,gv)] \\ &< \frac{11}{12}\max\left\{\left(x-\frac{x^2}{4}\right),\left(y-\frac{y^2}{4}\right)\right\}[d(gx,gu)+d(gy,gv)] \\ &\le \varphi(d(gx,gu)+d(gy,gv))[d(gx,gu)+d(gy,gv)]. \end{aligned}$$

Thus, for x, y ∈ [0,1] with $x, y \neq \frac{15}{32}$, the conditions (2.15) and (2.16) are satisfied. Following similar arguments, one can easily show that conditions (2.15) and (2.16) are also satisfied for $x \in [0, \frac{15}{32}) \cup (\frac{15}{32}, 1]$ and $y = \frac{15}{32}$. Finally, for $x = y = \frac{15}{32}$, if we assume that $gu = gv = \frac{15}{96}$, it follows that $d(gx, gu) + d(gy, gv) = \frac{15}{24}$.

Consequently, we get

$$\begin{aligned}\sqrt{\varphi(d(gx,gu)+d(gy,gv))}[d(gx,gu)+d(gy,gv)] &= \sqrt{\frac{11}{24}\cdot\frac{15}{24}}\cdot\frac{15}{24}\\ &< \frac{43}{80} = D(gx,F(x,y)) + D(gy,F(y,x))\end{aligned}$$

and

$$\begin{aligned}D(gu,F(u,v)) + D(gv,F(v,u)) &= 2\left|\frac{15}{96} - \frac{1}{4}\left(\frac{15}{96}\right)^2\right|\\ &< \frac{11}{12}\cdot\frac{15}{24}\cdot\frac{15}{24}\\ &= \varphi(d(gx,gu)+d(gy,gv))[d(gx,gu)+d(gy,gv)].\end{aligned}$$

Thus, we conclude that all the conditions of Theorem 2.2 are satisfied, and F, g admits a coupled coincidence point $z = (0, 0)$.

3. Coupled coincidences by mixed *g*-monotone property

Recently, there have been exciting developments in the field of existence of fixed points in partially ordered metric spaces (cf. [13-24]). Using the concept of commuting maps and mixed g-monotone property, Lakshmikantham and Ćirić in [5] established the existence of coupled coincidence point results to generalize the results of Bhaskar and Lakshmikantham [4]. Choudhury and Kundu generalized these results to compatible maps. In this section, we shall extend the concepts of commuting, compatible maps and mixed g-monotone property to the case when F is multi-valued map and prove the extension of the above mentioned results.

Analogous with mixed monotone property, Lakshmikantham and Ćirić [5] introduced the following concept of a mixed g-monotone property.

Definition 3.1. *Let* $(X, \preccurlyeq)$ *be a partially ordered set and* $F: X \times X \to X$ *and* $g: X \to X$. We say F has the *mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any* $x, y \in X$,

$$x_1, x_2 \in X, g(x_1) \preccurlyeq g(x_2) \quad implies \quad F(x_1, y) \preccurlyeq F(x_2, y) \tag{3.1}$$

and

$$y_1, y_2 \in X, g(y_1) \preccurlyeq g(y_2) \quad implies \quad F(x, y_1) \succcurlyeq F(x, y_2). \tag{3.2}$$

Definition 3.2. *Let* $(X, \preccurlyeq)$ *be a partially ordered set,* $F: X \times X \to CL(X)$ *and let* $g: X \to X$ *be a mapping. We say that the mapping F has the mixed g-monotone property if, for all* x_1, x_2, y_1, $y_2 \in X$ with $gx_1 \preccurlyeq gx_2$ *and* $gy_1 \succcurlyeq gy_2$, *we get for all* $gu_1 \in F(x_1, y_1)$ *there exists* $gu_2 \in F(x_2, y_2)$ *such that* $gu_1 \preccurlyeq gu_2$ *and for all* $gv_1 \in F(y_1, x_1)$ *there exists* $gv_2 \in F(y_2, x_2)$ *such that* $gv_1 \succcurlyeq z\ gv_2$.

Definition 3.3. *The mapping $F: X \times X \to CB(X)$ and $g: X \to X$ are said to be compatible if*

$$\lim_{n\to\infty} H(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0$$

and

$$\lim_{n\to\infty} H(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that $x = \lim_{n\to\infty} gx_n \in \lim_{n\to\infty} F(x_n, y_n)$ and $y = \lim_{n\to\infty} gy_n \in \lim_{n\to\infty} F(y_n, x_n)$, for all $x, y \in X$ are satisfied.

Definition 3.4. *The mapping $F: X \times X \to CB(X)$ and $g: X \to X$ are said to be commuting if $gF(x, y) \subseteq F(gx, gy)$ for all $x, y \in X$.*

Lemma 3.1. [1]*If $A,B \in CB(X)$ with $H(A, B) < \epsilon$, then for each $a \in A$ there exists an element $b \in B$ such that $d(a, b) < \epsilon$.*

Lemma 3.2. [1]*Let $\{A_n\}$ be a sequence in $CB(X)$ and $\lim_{n\to\infty} H(A_n, A) = 0$ for $A \in CB(X)$. If $x_n \in A_n$ and $\lim_{n\to\infty} d(x_n, x) = 0$, then $x \in A$.*

Let $(X, \preccurlyeq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. We define the partial order on the product space $X \times X$ as:

for $(u,v),(x,y) \in X \times X$, $(u, v) \preccurlyeq (x, y)$ if and only if $u \preccurlyeq x$, $v \succcurlyeq y$.

The product metric on $X \times X$ is defined as

$$d((x_1, y_1), (x_2, y_2)) := d(x_1, x_2) + d(y_1, y_2) \quad \text{for all } x_i, y_i \in X (i = 1, 2).$$

For notational convenience, we use the same symbol d for the product metric as well as for the metric on X.

We begin with the following result that gives the existence of a coupled coincidence point for compatible maps F and g in partially ordered metric spaces, where F is the multi-valued mappings.

Theorem 3.1. *Let $F: X \times X \to CB(X)$, $g: X \to X$ be such that:*

(1) *there exists $\kappa \in (0,1)$ with*

$$H(F(x, y), F(u, v)) \le \frac{k}{2} d((gx, gy), (gu, gv)) \quad \textit{for all } (gx, gy) \succcurlyeq (gu, gv);$$

(2) *if $gx_1 \preccurlyeq gx_2$, $gy_2 \preccurlyeq gy_1$, $x_i, y_i \in X (i = 1,2)$, then for all $gu_1 \in F(x_1, y_1)$ there exists $gu_2 \in F(x_2, y_2)$ with $gu_1 \preccurlyeq gu_2$ and for all $gv_1 \in F(y_1, x_1)$ there exists $gv_2 \in F(y_2, x_2)$ with $gv_2 \preccurlyeq gv_1$ provided $d((gu_1, gv_1), (gu_2, gv_2)) < 1$; i.e. F has the mixed g-monotone property, provided $d((gu_1, gv_1), (gu_2, gv_2)) < 1$;*

(3) there *exists $x_0, y_0 \in X$, and some $gx_1 \in F(x_0, y_0)$, $gy_1 \in F(y_0, x_0)$ with $gx_0 \preccurlyeq gx_1$, $gy_0 \succcurlyeq gy_1$ such that $d((gx_0, gy_0), (gx_1, gy_1)) < 1 - \kappa$, where $\kappa \in (0,1)$;*

(4) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \le x$ for all n and if a non-increasing sequence $\{y_n\} \to y$, then $y \le y_n$ for all n and gX is complete.*

Then, F and g have a coupled coincidence point.

Proof. Let $x_0, y_0 \in X$ then by (3) there exists $gx_1 \in F(x_0, y_0)$, $gy_1 \in F(y_0, x_0)$ with $gx_0 \preccurlyeq gx_1$, $gy_0 \succcurlyeq gy_1$ such that

$$d((gx_0, gy_0), (gx_1, gy_1)) < 1 - \kappa. \tag{3.3}$$

Since $(gx_0, gy_0) \preccurlyeq (gx_1, gy_1)$ using (1) and (3.3), we have

$$H(F(x_0, y_0), F(x_1, y_1)) \leq \frac{\kappa}{2} d((gx_0, gy_0), (gx_1, gy_1)) < \frac{\kappa}{2}(1 - \kappa)$$

and similarly

$$H(F(y_0, x_0), F(y_1, x_1)) \leq \frac{\kappa}{2}(1 - \kappa).$$

Using (2) and Lemma 3.1, we have the existence of $gx_2 \in F(x_1, y_1)$, $gy_2 \in F(y_1, x_1)$ with $x_1 \preccurlyeq x_2$ and $y_1 \succcurlyeq y_2$ such that

$$d(gx_1, gx_2) \leq \frac{\kappa}{2}(1 - \kappa) \tag{3.4}$$

and

$$d(gy_1, gy_2) \leq \frac{\kappa}{2}(1 - \kappa). \tag{3.5}$$

From (3.4) and (3.5),

$$d((gx_1, gy_1), (gx_2, gy_2)) \leq \kappa(1 - \kappa). \tag{3.6}$$

Again by (1) and (3.6), we have

$$H(F(x_1, y_1), F(x_2, y_2)) \leq \frac{\kappa^2}{2}(1 - \kappa)$$

and

$$D(F(y_1, x_1), F(y_2, x_2)) \leq \frac{\kappa^2}{2}(1 - \kappa).$$

From Lemma 3.1 and (2), we have the existence of $gx_3 \in F(x_2, y_2)$, $gy_3 \in F(y_2, x_2)$ with $gx_2 \preccurlyeq gx_3$, $gy_2 \succcurlyeq gy_3$ such that

$$d(gx_2, gx_3) \leq \frac{\kappa^2}{2}(1 - \kappa)$$

and

$$d(gy_2, gy_3) \leq \frac{\kappa^2}{2}(1 - \kappa).$$

It follows that

$$d((gx_2, gy_2), (gx_3, gy_3)) \leq \kappa^2(1 - \kappa).$$

Continuing in this way we obtain $gx_{n+1} \in F(x_n, y_n)$, $gy_{n+1} \in F(y_n, x_n)$ with $gx_n \preccurlyeq gx_{n+1}, gy_n \succcurlyeq gy_{n_1}$ such that

$$d(gx_n, gx_{n+1}) \leq \frac{\kappa^n}{2}(1-\kappa)$$

and

$$d(gy_n, gy_{n+1}) \leq \frac{\kappa^n}{2}(1-\kappa).$$

Thus,

$$d((gx_n, gy_n), (gx_{n+1}, gy_{n+1})) \leq \kappa^n(1-\kappa). \tag{3.7}$$

Next, we will show that $\{gx_n\}$ is a Cauchy sequence in X. Let $m > n$. Then,

$$\begin{aligned} d(gx_n, gx_m) &\leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + d(gx_{n+2}, gx_{n+3}) + \cdots + d(gx_{m-1}, gx_m) \\ &\leq [\kappa^n + \kappa^{n+1} + \kappa^{n+2} + \cdots + \kappa^{m-1}]\frac{(1-\kappa)}{2} \\ &= \kappa^n[1 + \kappa + \kappa^2 + \cdots + \kappa^{m-n-1}]\frac{(1-\kappa)}{2} \\ &= \kappa^n\left[\frac{1-\kappa^{m-n}}{1-\kappa}\right]\frac{(1-\kappa)}{2} \\ &= \frac{\kappa^n}{2}(1-\kappa^{m-n}) < \frac{\kappa^n}{2}, \end{aligned}$$

because $\kappa \in (0,1)$, $1 - \kappa^{m-n} < 1$. Therefore, $d(gx_n, gx_m) \to 0$ as $n \to \infty$ implies that $\{gx_n\}$ is a Cauchy sequence. Similarly, we can show that $\{gy_n\}$ is also a Cauchy sequence in X. Since gX is complete, there exists $x, y \in X$ such that $gx_n \to gx$ and $gy_n \to gy$ as $n \to \infty$. Finally, we have to show that $gx \in F(x, y)$ and $gy \in F(y, x)$.

Since $\{gx_n\}$ is a non-decreasing sequence and $\{gy_n\}$ is a non-increasing sequence in X such that $gx_n \to x$ and $gy_n \to y$ as $n \to \infty$, therefore we have $gx_n \preccurlyeq x$ and $gy_n \succcurlyeq y$ for all n. As $n \to \infty$, (1) implies that

$$H(F(x_n, y_n), F(x, y)) \leq \frac{\kappa}{2}d((gx_n, gy_n), (gx, gy)) \to 0.$$

Since $gx_{n+1} \in F(x_n, y_n)$ and $\lim_{n\to\infty} d(gx_{n+1}, gx) = 0$, it follows using Lemma 3.2 that $gx \in F(x, y)$. Again by (1),

$$H(F(y_n, x_n), F(y, x)) \leq \frac{\kappa}{2}d((gy_n, gx_n), (gy, gx)) \to 0.$$

Since $gy_{n+1} \in F(y_n, x_n)$ and $\lim_{n\to\infty} d(gy_{n+1}, gy) = 0$, it follows using Lemma 3.2 that $gy \in F(y, x)$.

Theorem 3.2. *Let F: $X \times X \to CB(X)$, g: $X \to X$ be such that conditions (1)-(3) of Theorem 3.1 hold. Let X be complete, F and g be continuous and compatible. Then, F and g have a coupled coincidence point.*

Proof. As in the proof of Theorem 3.1, we obtain the Cauchy sequences $\{gx_n\}$ and $\{gy_n\}$ in X. Since X is complete, there exists $x, y \in X$ such that $gx_n \to x$ and $gy_n \to y$ as $n \to \infty$. Finally, we have to show that $gx \in F(x, y)$ and $gy \in F(y, x)$. Since the mapping F: $X \times X \to CB\ (X)$ and g: $X \to X$ are compatible, we have

$$\lim_{n\to\infty} H(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0,$$

because $\{x_n\}$ is a sequence in X, such that $x = \lim_{n\to\infty} gx_{n+1} \in \lim_{n\to\infty} F(x_n, y_n)$ is satisfied. For all $n \geq 0$, we have

$$D(gx, F(gx_n, gy_n)) \leq D(gx, gF(x_n, y_n)) + H(gF(x_n, y_n), F(gx_n, gy_n)).$$

Taking the limit as $n \to \infty$, and using the fact that g and F are continuous, we get, $D(gx, F(x, y)) = 0$, which implies that $gx \in F(x, y)$.

Similarly, since the mapping F and g are compatible, we have

$$\lim_{n\to\infty} H(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0,$$

because $\{y_n\}$ is a sequence in X, such that $y = \lim_{n\to\infty} gy_{n+1} \in \lim_{n\to\infty} F(y_n, x_n)$ is satisfied. For all $n \geq 0$, we have

$$D(gy, F(gy_n, gx_n)) \leq D(gy, gF(y_n, x_n)) + H(gF(y_n, x_n), F(gy_n, gx_n)).$$

Taking the limit as n $\to \infty$, and using the fact that g and F are continuous, we get $D(gy, F(y, x)) = 0$, which implies that $gy \in F(y, x)$.

As commuting maps are compatible, we obtain the following;

Theorem 3.3. *Let F: $X \times X \to CB(X)$, g: $X \to X$ be such that conditions (1)-(3) of Theorem 3.1 hold. Let X be complete, F and g be continuous and commuting. Then, F and g have a coupled coincidence point.*

Authors' contributions

The authors have equitably contributed in obtaining the new results presented in this article. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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